

# Self-gravitating charged fluid spheres with anisotropic pressures

Ambrish M. Raghoonundun · David W. Hobill

4 March 2016

**Abstract** The coupled system of the spherically symmetric Einstein–Maxwell differential equations is solved under two different source conditions: non-zero electric charge and pressure anisotropy. Expressions for the metric functions, and pressures which extend the Tolman VII exact solution are deduced from the new solutions. By applying boundary conditions to these solutions, all integration constants are computed in terms of parameters that are physically meaningful.

**Keywords** Einstein equations, Exact Solutions, Electric Charge, Pressure Anisotropy, Compact Objects

**PACS** 04.20.-q, 04.40.Nr, 26.60.Kp

## 1 Introduction

The Tolman VII solution [15] is an exact analytic solution to the static and spherically symmetric Einstein’s field equations (EFE), with a perfect fluid matter source. This solution obeys all the criteria for physical acceptability [4, 6], and has been shown to be a viable solution that can be used for modelling compact objects such as neutron and quark stars [13].

In this article, that solution is generalized to generate new exact solutions with charge and/or anisotropic pressures. Since the Tolman VII solution provides a physically valid model of various physical objects, the hope behind these generalizations is that they will lead to new models that share these characteristics.

---

Ambrish M. Raghoonundun  
University of Calgary, 2500 University Dr. NW, Calgary, Alberta, Canada, T2N 1N4,  
E-mail: amraghoo@ucalgary.ca

David W. Hobill  
University of Calgary, 2500 University Dr. NW, Calgary, Alberta, Canada, T2N 1N4,  
E-mail: hobilll@ucalgary.ca

First a particular uncharged generalization of the Tolman VII solution is carried out by introducing an anisotropic pressure. The case for anisotropic pressures was made some time ago in Refs. [7, 2] and since then a number of studies into their properties have been published. Of note is the work of Letelier [10] who elucidates the possible physical interpretations of anisotropic pressures, and concludes that a mixture of a number of perfect fluids can be transformed, through a coordinate change, into *one* anisotropic fluid. This aspect simplifies the physical interpretation of anisotropic pressure models, and leads one to believe that anisotropic solutions can be candidates for simple physical models of gravitating fluid spheres.

Under the assumption of pressure anisotropy, the components of the pressure, which are assumed to be the same in all directions in a perfect fluid, must now be generalized to two different functions, which for intuitive reasons are usually called  $p_r$  for the radial pressure component, and  $p_\perp$  for the angular pressure component.

The further generalization carried out in this article is the introduction of electric charge in the models. While the existence of electrically charged, large scale physical objects are unlikely [3], on time scales much shorter than the whole lifetime of the compact object, charge could account for non-permanent properties such as magnetism (for example to model magnetars) in these objects.

Furthermore, as has been noted by numerous authors [8, 9, 16], in the static limit, the addition of charge does not change the difficulty of solving the EFE, since a “Maxwell differential equation” for the electric charge is added to the system of equations to be solved. This can immediately be integrated and incorporated into a global charge that is seen from the outside only through the Reissner-Nordström external metric. The EFE’s do not change drastically either, and a similar solution procedure to the one usually employed for the Tolman solutions can be used to great effect.

As a guide to both old and new solutions, we provide an overview in the form of a chart in Figure 1. This chart is organized so that when moving from left to right, one moves from solutions with isotropic pressures to those with anisotropic pressures in the middle to solutions with both charge and anisotropic pressures on the extreme right. This Figure uses quantities that are explained throughout the article. Boxes that form an end point in the decision flow are solutions that are either previously known (references given) or are new and discussed in the section indicated.

This article is organized as follows. Section 2 introduces the system of equations to be solved, and goes through the process of solving the system, while providing physical motivations for the various assumptions required for the solution. Section 3 concentrates on uncharged anisotropic pressures, and Section 4 constructs new solutions which have electric charge. The last section reviews the methods and discusses further research to be carried out on the physical relevance of the solutions.

## 2 The ODE system, method of solution, and physical considerations

### 2.1 The ODE system

In what follows, the metric is assumed to be spherically symmetric, static, and expressed in the usual Schwarzschild spherical coordinates  $(t, r, \theta, \varphi)$  as

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (1)$$

Throughout this derivation, an equivalent re-parametrization of the metric coefficients is also used, so that  $Z(r) = e^{-\lambda(r)}$  and  $Y(r) = e^{\nu(r)/2}$ . This re-parametrization will allow the differential equations to be simplified and depending on the type of matter considered in the following sections, the EFE can be reduced to linear ODEs for  $Z(r)$  and  $Y(r)$ .

The matter content is specified by the form of the energy momentum tensor. The spherically symmetric fluid energy-momentum tensor with anisotropic pressure is given by

$$T^i_j = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p_r & 0 & 0 \\ 0 & 0 & -p_\perp & 0 \\ 0 & 0 & 0 & -p_\perp \end{pmatrix}, \quad (2)$$

with  $\rho, p_r$  and  $p_\perp$  the matter density, radial pressure, and angular pressure, respectively.

The addition of electric charge to this system is made through the Faraday tensor, which in the static case is given [11, 3] by

$$F_{ab} = \begin{pmatrix} 0 & -\frac{qY}{r^2\sqrt{Z}} & 0 & 0 \\ \frac{qY}{r^2\sqrt{Z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

Here  $q(r)$  is the charge enclosed, with  $Y$  and  $Z$  the metric functions previously defined. The electromagnetic energy momentum tensor is of the form

$$T_{ab}^{\text{EM}} = g_{ac} F^{cd} F_{db} - \frac{1}{4} g_{ab} F^{cd} F_{cd},$$

and together with (3) and (2), the total energy momentum tensor for the fluid and the electric field source is given by [8]:

$$T^i_j = \begin{pmatrix} \rho + \frac{q^2}{\kappa r^4} & 0 & 0 & 0 \\ 0 & -p_r + \frac{q^2}{\kappa r^4} & 0 & 0 \\ 0 & 0 & -p_\perp - \frac{q^2}{\kappa r^4} & 0 \\ 0 & 0 & 0 & -p_\perp - \frac{q^2}{\kappa r^4} \end{pmatrix}. \quad (4)$$

As a result, the complete Einstein-Maxwell system (EMS) of field equations becomes

$$\kappa\rho + \frac{q^2}{r^4} = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{1}{r^2} - \frac{Z}{r^2} - \frac{1}{r} \frac{dZ}{dr}, \quad (5a)$$

$$\kappa p_r - \frac{q^2}{r^4} = e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{2Z}{rY} \frac{dY}{dr} + \frac{Z}{r^2} - \frac{1}{r^2}, \quad (5b)$$

$$\begin{aligned} \kappa p_\perp + \frac{q^2}{r^4} &= e^{-\lambda} \left( \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} \right) \\ &= \frac{Z}{Y} \frac{d^2Y}{dr^2} + \frac{1}{2Y} \frac{dY}{dr} \frac{dZ}{dr} + \frac{Z}{rY} \frac{dY}{dr} + \frac{1}{2r} \frac{dZ}{dr}, \end{aligned} \quad (5c)$$

$$F_{10} = -F_{01} = \frac{q}{r^2} e^{(\nu+\lambda)/2} = \frac{qY}{r^2 \sqrt{Z}}, \quad (5d)$$

The solution of this set of equations can be constructed in a variety of ways, and for an extensive review, see Ref. [8]. However the aim of this article is to find new solutions, that can be considered to be physically relevant and therefore we proceed with a method outlined in [15].

## 2.2 Solution method

First we assume a functional form for the  $Z$  metric function given by

$$Z(r) = 1 - br^2 + ar^4. \quad (6)$$

For the moment,  $a$  and  $b$  are undetermined constants, and  $r$  is the radial coordinate. The physical interpretation of the constants will come later in Section 2.3.

This assumption helps in solving the system of equation (5) for  $Z$ , since the first order equation (5a) can immediately be integrated in terms of one integration constant. To find the solution for  $Y$ , one first defines a new quantity, “the measure of anisotropy,”  $\Delta$  through subtraction of (5c) from (5b),

$$\begin{aligned} \kappa\Delta = \kappa(p_r - p_\perp) &= \frac{Z}{rY} \left( \frac{dY}{dr} \right) - \frac{Z}{Y} \left( \frac{d^2Y}{dr^2} \right) - \frac{1}{2Y} \left( \frac{dZ}{dr} \right) \left( \frac{dY}{dr} \right) \\ &\quad - \frac{1}{2r} \left( \frac{dZ}{dr} \right) + \frac{Z}{r^2} - \frac{1}{r^2} + \frac{2q^2}{r^4}. \end{aligned} \quad (7)$$

This equation can be rearranged and simplified as a linear second order ODE for  $Y$ , which can then be solved using a transformation of the independent variable  $r$ .

$$2r^2 Z \left( \frac{d^2Y}{dr^2} \right) + \left[ r^2 \left( \frac{dZ}{dr} \right) - 2rZ \right] \frac{dY}{dr} +$$

$$+ \left( 2 + 2r^2 \Delta - 2Z + r \frac{dZ}{dr} - \frac{4q^2}{r^2} \right) Y = 0. \quad (8)$$

The second order ODE has both  $\Delta$  and  $q$  as undetermined functions, which when set to zero transforms the ODE into that for the Tolman VII solution discussed in detail in [13]. Therefore this procedure provides a generalization of the Tolman VII solution.

The next step in the solution is the transformation to a new variable  $x = r^2$ . A straight forward derivation then transforms the derivative

$$\frac{d}{dr} \equiv 2\sqrt{x} \frac{d}{dx},$$

and similarly

$$\frac{d^2}{dr^2} \equiv 4x \frac{d^2}{dx^2} + 2 \frac{d}{dx}.$$

Applying these to equation (8) results in

$$2xZ \left( 4x \frac{d^2 Y}{dx^2} + 2 \frac{dY}{dx} \right) + \left( 2\sqrt{x} \frac{dY}{dx} \right) \left( 2x^{3/2} \frac{dZ}{dx} - 2\sqrt{x} Z \right) + \left[ 2 + 2x\Delta + \sqrt{x} \left( 2\sqrt{x} \frac{dZ}{dx} \right) - 2Z - \frac{4q^2}{x} \right] Y = 0,$$

which can be rearranged into

$$8x^2 Z \frac{d^2 Y}{dx^2} + \left( 4xZ + 4x^2 \frac{dZ}{dx} - 4xZ \right) \frac{dY}{dx} + 2 \left( 1 + x\Delta + x \frac{dZ}{dx} - Z - \frac{2q^2}{x} \right) Y = 0,$$

or equivalently,

$$Z \frac{d^2 Y}{dx^2} + \left( \frac{1}{2} \frac{dZ}{dx} \right) \frac{dY}{dx} + \left( \frac{1 + x\Delta + x \frac{dZ}{dx} - Z - 2q^2/x}{4x^2} \right) Y = 0. \quad (10)$$

The second step of the solution procedure involves another radial variable change from  $x$  to  $\xi$  which is defined through

$$\xi = \int_0^x \frac{d\bar{x}}{\sqrt{Z(\bar{x})}} = \frac{2}{\sqrt{a}} \operatorname{arccoth} \left( \frac{1 + \sqrt{1 - bx + ax^2}}{\sqrt{a}x} \right). \quad (11)$$

This induces a change in the  $x$ -derivatives, so that

$$\frac{d}{dx} \equiv \frac{1}{\sqrt{Z(x)}} \frac{d}{d\xi}, \quad \text{and}, \quad (12a)$$

$$\frac{d^2}{dx^2} \equiv \frac{1}{Z} \frac{d^2}{d\xi^2} - \frac{1}{2Z^{3/2}} \frac{dZ}{dx} \frac{d}{d\xi}. \quad (12b)$$

Applying these changes to the differential equation (10) results in the elimination of the first derivative term for  $Y$ , further simplifying the second order ODE:

$$Z \left\{ \frac{1}{Z} \frac{d^2 Y}{d\xi^2} - \frac{1}{2Z^{3/2}} \frac{dZ}{dx} \frac{dY}{d\xi} \right\} + \frac{1}{2} \frac{dZ}{dx} \left( \frac{1}{\sqrt{Z}} \frac{dY}{d\xi} \right) + \left( \frac{1 + x\Delta + x \frac{dZ}{dx} - Z - 2q^2/x}{4x^2} \right) Y = 0. \quad (13)$$

Since  $Z$  is a known function of  $x$ , simplification of the coefficient multiplying  $Y$ , is immediately possible. This reduces equation (13) into a simple linear ODE since,

$$\left( \frac{1 + x\Delta + x(2ax - b) - (1 - bx + ax^2) - 2q^2/x}{4x^2} \right) = \left( \frac{a}{4} + \frac{\Delta}{4x} - \frac{q^2}{2x^3} \right),$$

so that the ODE for  $Y$  finally becomes

$$\frac{d^2 Y}{d\xi^2} + \left( \frac{a}{4} + \frac{\Delta}{4x} - \frac{q^2}{2x^3} \right) Y = 0. \quad (14)$$

When the coefficient multiplying  $Y$  is a constant, simple closed form solutions are easily obtained. As mentioned previously,  $\Delta$  is a function that can be arbitrarily chosen, and is interpreted as a measure of anisotropy between the pressures in the solution. From spherical symmetry, both the radial pressure  $p_r$  and the tangential pressure  $p_\perp$  must be equal at the centre, resulting in  $\Delta$  having to be equal to zero when  $x = r^2 = 0$ . Similarly the enclosed charge  $q$  can be given arbitrarily. In order to be physically relevant  $q$  must be regular everywhere including at the origin where  $r = 0$ .

### 2.3 Physical considerations

Following [13], where a straightforward interpretation of the Tolman VII solution in terms of physical parameters of the star was given, a physical motivation for the different functions in the equation (14) above is sought. This section can be seen as the motivation for choosing the radial dependence for  $\Delta$  and  $q$ . Considering that in the Tolman VII solution, the ansatz for  $Z$  allows the integration of equation (5a) directly, proceeding in a similar fashion here simplifies the ODE (5a), with the caveat that now there is a contribution from the charge  $q$ .

However, in general relativity, the electric charge has mass-energy, and following Ivanov [8] the external perceived mass is redefined as the sum of the material rest mass, and the electrostatic energy contained in the electrostatic field, so that

$$M = 4\pi \int_0^{r_b} \left( \rho(r) + \frac{q^2(r)}{8\pi r^4} \right) r^2 dr. \quad (15)$$

This redefinition of mass then results in the integration of equation (5a) as follows, since  $Z$  has already an assumed form,

$$M = 4\pi \int_0^{r_b} \left( 1 - Z - r \frac{dZ}{dr} \right) dr = 4\pi \int_0^{r_b} (3br^2 - 5ar^4) dr. \quad (16)$$

This mass equation leads to a physical interpretation for the constants  $a$  and  $b$ . Since the Tolman VII solution has physical characteristics, *and* is compatible with the form of  $Z$  chosen, one assumes the same form in all of the future solutions. The physical reasoning motivating this choice is given in detail in [13], where a quadratic density profile of the form

$$\rho = \rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right], \quad (17)$$

leads to a metric function  $Z$  becomes

$$Z(r) = 1 - \left( \frac{\kappa \rho_c}{3} \right) r^2 + \left( \frac{\kappa \mu \rho_c}{5 r_b^2} \right) r^4 = 1 - b r^2 + a r^4. \quad (18)$$

Thus  $a$  and  $b$  can be identified with the parameters in parentheses. In (17), the constant  $r_b$  represents the boundary radius at the matter-vacuum interface, and  $\rho_c$  represents the central density at  $r = 0$ . Finally  $\mu$  is a dimensionless “self-boundedness” parameter where  $0 \leq \mu \leq 1$ . When  $\mu = 0$ , one has a sphere of constant density, and when  $\mu = 1$ , a “natural” star, with density vanishing at the boundary is obtained. For intermediate values there is a density discontinuity at  $r_b$  which decreases in magnitude with increasing  $\mu$ .

This same property shall be used once again in Section 3, where an anisotropic pressure only is considered. For the case with charge a more general approach will be taken.

The charge distribution will be assumed to take the form  $q(r) = k r^n$ , where  $k$  is a constant and  $n > 0$ . Such charge distributions occur in many static situations, for mathematical reasons (see [8] for example) but they can be expected to exist on physical grounds as well.

First all known charge carriers are massive. Therefore they will also undergo gravitational attraction. In addition the static electromagnetic field energy density is positive since it is proportional to  $q^2$  which then also contributes to the attractive gravitational force. On the other hand it is known that in the absence of any other forces all free charges reside on the surface of a conducting medium.

It can therefore be expected that when gravitational and electromagnetic forces are closed to being balanced, the strength of the electromagnetic repulsion will tend to concentrate charge in the outer regions of a compact fluid body, while maintaining a non-zero charge density inside the body.

The fact that pressure terms are introduced due to non-gravitational, non-electromagnetic interactions helps provide more realistic charge distributions than are found in charged dust solutions for example.

From the above discussion the initial ansatz for  $Z$  from equation (6) can be substituted into the RHS of the first ODE (5a), which results in

$$\kappa \rho + \frac{q^2}{r^4} = 3b - 5ar^2. \quad (19)$$

Consistency, and the desire to keep the procedure for solving this system of equations the same as outlined in Section 2.2 then demands that the LHS of the

equation (19) also be a quadratic function with vanishing linear term. Because of the structure of this equation one is forced to choose either  $q(r) = kr^2$ , in which case,

$$3b - 5ar^2 = (\kappa\rho_c + k^2) - \frac{\kappa\rho_c\mu}{r_b^2}r^2,$$

or  $q(r) = kr^3$ , which results in

$$3b - 5ar^2 = \kappa\rho_c - \left(\frac{\kappa\rho_c\mu}{r_b^2} - k^2\right)r^2.$$

Therefore the charge introduces terms that redefine the parameters  $a$  and  $b$ , in terms of physical parameters  $\rho_c, \mu, r_b$  and  $k$ . However, it is found that the choice  $q(r) = kr^2$  yields a differential equation for  $Y(r)$  that is not soluble with elementary functions<sup>1</sup>. Discarding the quadratic radial dependence, the choice  $q(r) = kr^3$  is selected for the remainder of this article.

Turning to the “measure of anisotropy”  $\Delta(r)$ , one has to satisfy the condition that the metric is spherically symmetric. Both pressures *must* be equal at the coordinate centre,  $r = 0$  if spherical symmetry is to be satisfied, and this can be translated into  $\Delta(0) = 0$ . Except for this condition, there is complete arbitrariness in specifying the anisotropy, and indeed the literature is replete with different ansätze for  $\Delta$ . For the purely mathematical reason of solving equation (14) the choice we make is  $\Delta = \beta x = \beta r^2$ . This choice satisfies the condition at the centre, *and* leads to a constant value for the coefficient of  $Y$  appearing equation (14).

With these two assumptions about the anisotropy and charge, the ODE for  $Y$  becomes

$$\frac{d^2Y}{d\xi^2} = -\frac{Y}{4}(a + \beta - 2k^2) = -\Phi^2 Y. \quad (20)$$

Different solutions can be generated from different assumptions about  $\Phi^2, \beta$  or  $k$ , and a classification that summarizes these choices is provided in Figure 1. Those in the lightly shaded boxes are the ones that will be discussed in this article.

## 2.4 Boundary conditions

The constants of integration associated with each solution will be determined by the boundary conditions (BCs) that one specifies. The system of equations consists of one first order ODE for  $Z$ , and one second order ODE for  $Y$ . The first one makes use of the definition of mass in equation (15). For the second one, we require two BCs, and the junction conditions matching the interior metric to the exterior metric provide the two needed BCs.

<sup>1</sup> The coefficient of  $Y$  in the second order ODE after variable changes still contains a  $1/r^2$  term, turning the problem into a variable coefficient one. Once additional assumptions about  $a$  have been made, a solution in terms of hypergeometric functions is possible, but the assumption about  $a$  renders the solution physically uninteresting.



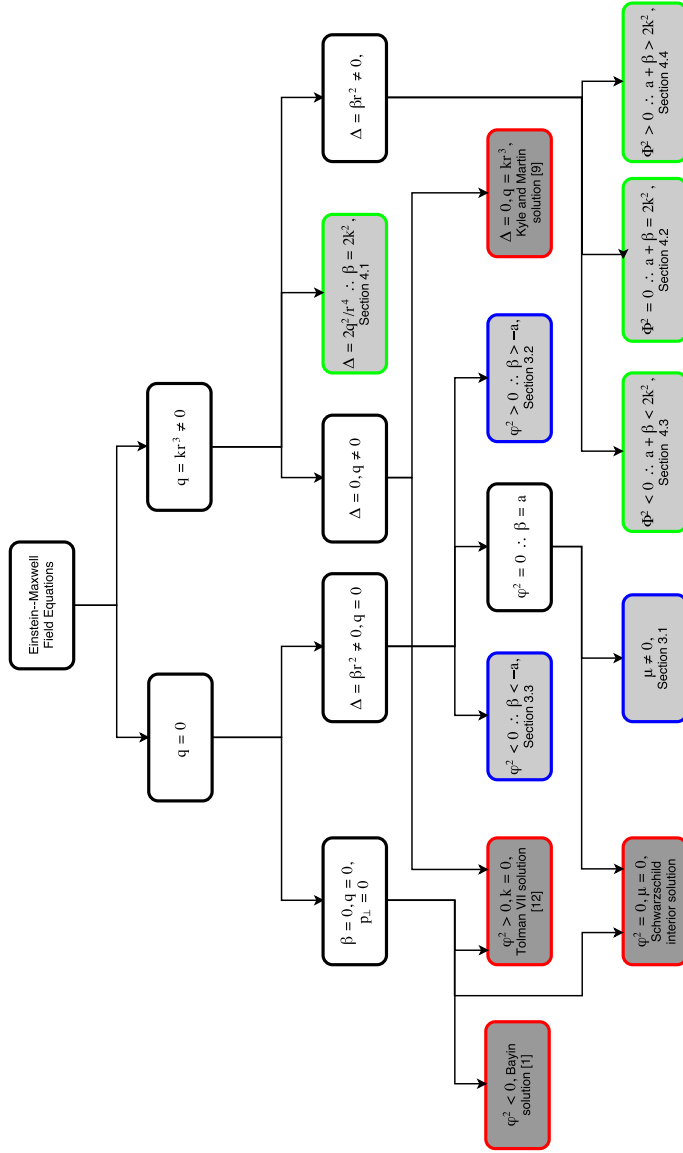


Fig. 1: The solution landscape explored in this article. Lightly shaded boxes are the new solutions described in this work, and darker ones are the older known solutions. In the online coloured version, red-bordered boxes are solutions with isotropic pressure, blue-bordered ones are uncharged solutions with anisotropic pressures only and green-bordered ones are charged with anisotropic pressure.

These junction conditions in the case of spherically symmetric and static solutions, match the interior solution to the charged exterior Reissner–Nordström metric given by

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (21)$$

where  $M$  is the mass given by (15) and  $Q$  the charge given by

$$Q = q(r_b).$$

The radial pressure which can be computed once the solution for  $Y$  has been specified, has to vanish at the boundary of the compact object where the exterior and interior metric meet. Therefore

$$p_r(r_b) = 0, \quad \text{and}, \quad (22a)$$

$$Z(r_b) = 1 - \frac{2M}{r_b} + \frac{Q^2}{r_b^2} = Y^2(r_b). \quad (22b)$$

The second boundary condition (22b) can also be implemented once the solution for  $Y$  has been specified. It is possible to devise an expression for the radial pressure  $p_r$  solely in terms of the metric and metric derivatives that make applying the first boundary condition easier. The first step in generating this expression is the addition of equations (5a) and (5a) to obtain

$$\kappa(p_r + \rho) = e^{-\lambda} \left( \frac{\nu'}{r} + \frac{\lambda'}{r} \right) = \frac{2Z}{rY} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr}. \quad (23)$$

Going through the same coordinate changes as discussed in Section 2.2 one obtains

$$\kappa p_r = \frac{2Z}{rY} \frac{dY}{dr} - \frac{1}{r} \frac{dZ}{dr} - \kappa \rho \xrightarrow{r \rightarrow x} \frac{4Z}{Y} \frac{dY}{dx} - 2 \frac{dZ}{dx} - \kappa \rho \xrightarrow{x \rightarrow \xi} \frac{4\sqrt{Z}}{Y} \frac{dY}{d\xi} - 2 \frac{dZ}{dx} - \kappa \rho.$$

This expression can be evaluated at the boundary  $r = r_b$ , with conditions (22) simplifying the results,

$$\kappa p_r(r_b) = 0 = \frac{4\sqrt{Z(r_b)}}{Y(r_b)} \frac{dY}{d\xi} \Big|_{\xi=\xi_b} - 2 \frac{dZ}{dx} \Big|_{x=x_b} - \kappa \rho(r_b),$$

so that

$$\kappa \rho(r_b) = 4 \frac{dY}{d\xi} \Big|_{\xi=\xi_b} - 2 \frac{dZ}{dx} \Big|_{x=x_b}.$$

This leads to a general boundary condition on the derivative of  $Y$

$$\frac{dY}{d\xi} \Big|_{\xi=\xi_b} = \frac{1}{4} \left[ \frac{\kappa \rho_c}{3} - \frac{\kappa \rho_c \mu}{5} - \frac{4k^2 r_b^2}{5} \right] =: \alpha. \quad (24)$$

For the second condition one re-expresses equation (22b) in terms of  $Y$  through

$$Y(r_b) = \sqrt{Z(r_b)} = \sqrt{1 + \frac{\kappa \rho_c r_b^2 (3\mu - 5)}{15} - \frac{k^2 r_b^4}{5}} =: \gamma. \quad (25)$$

Subsequent application of these two conditions on the  $Y$  metric function form a Cauchy boundary pair and results in unique integration constants for the  $Y$  metric function in terms of the auxiliary constants  $\alpha$  and  $\gamma$ , defined in equations (24) and (25) respectively.

## 2.5 New Solutions

The spherically symmetric Einstein–Maxwell interior field equations can now be solved completely since a distinct ODE whose solutions are available has been presented, and furthermore, one can interpret all the constants present in the solutions physically. The availability of boundary conditions ensures the uniqueness of the solution. Based on Figure 1 the ODE for  $Y$  for different physical conditions will be discussed in the following sections.

## 3 Uncharged case with anisotropic pressures

In this section, only the uncharged  $q = k = 0$  specialization of the ODE (20) is investigated. As a result, this equation becomes

$$\frac{d^2 Y}{d\xi^2} + \left( \frac{a + \beta}{4} \right) Y = 0.$$

The solutions can immediately be written in terms of the parameter  $\phi^2 = \Phi^2(k = 0) = (a + \beta)/4$  in the following table. The details of the properties associated with the different solutions are discussed in the sections indicated in Table 1.

$\phi^2$	$Y(\xi)$	Solution's analysis
$\phi^2 < 0$	$c_1 \cosh \left( \sqrt{-\phi^2} \xi \right) + c_2 \sinh \left( -\sqrt{-\phi^2} \xi \right)$	section 3.3
$\phi^2 = 0$	$c_1 + c_2 \xi$	section 3.1
$\phi^2 > 0$	$c_1 \cos (\phi \xi) + c_2 \sin (\phi \xi)$	section 3.2

Table 1: The different solutions that can be generated through different values of the parameter  $\phi^2$ . The integration constants  $c_1$ , and  $c_2$  are determined by the two boundary conditions.

### 3.1 The $\phi^2 = 0$ case

When  $\phi^2 = 0$ , the only possibility is for  $\beta = -a = -\frac{\kappa\mu\rho_c}{5r_b^2}$ , which is either negative when all the parameters in the  $\frac{\kappa\mu\rho_c}{5r_b^2}$  expression are positive definite: the case we will consider now, or zero when the parameter  $\mu = 0$ . The latter case reduces to the Schwarzschild interior solution for which there is much historical [14, 18] and contemporary literature [17, 5]. For the  $\beta \neq 0$  case,  $p_\perp = p_r - \Delta = p_r + ax$ , and the angular pressure is thus larger than the radial pressure everywhere but at the centre. Applying the two boundary conditions to solve for the integration constants, equation (24) yields

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = c_2 = \alpha := \frac{1}{4} \left( \frac{\kappa\rho_c}{3} - \frac{\mu\kappa\rho_c}{5} \right), \quad (26)$$

where the  $\xi$ -derivative can be computed for  $Y$  from its expression given in Table 1. The constant  $\alpha$  can be obtained from expressions for the density and  $Z$  given in equation (17) and (18).

Using equation (25)

$$Y|_{\xi=\xi_b} = c_1 + c_2\xi_b = \gamma := \sqrt{Z(r_b)} = \sqrt{1 - \left(\frac{\kappa\rho_c}{3}\right)r_b^2 + \left(\frac{\kappa\mu\rho_c}{5r_b^2}\right)r_b^4}, \quad (27)$$

it can be deduced that

$$c_1 = \gamma - \frac{2\alpha}{\sqrt{a}} \operatorname{arccoth} \left( \frac{1+\gamma}{r_b^2\sqrt{a}} \right).$$

Figure 2 plots the metric functions and shows that the metric functions and their derivatives become equal at the radius  $r_b$ , as expected from the matching to the Schwarzschild exterior metric, at the boundary  $r_b$ .

The solution for the second metric function as a function of  $r$  can, after the changes in the radial parameter, be given as

$$Y(r) = \gamma + \frac{2\alpha r_b}{\sqrt{\kappa\rho_c\mu/5}} \left[ \operatorname{arccoth} \left( \frac{1 - \sqrt{Z(r)}}{r^2 \sqrt{\frac{\kappa\rho_c\mu}{5r_b^2}}} \right) - \operatorname{arccoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa\rho_c\mu/5}} \right) \right], \quad (28)$$

where the constants  $\alpha$  and  $\gamma$  are given in terms of the fundamental set of parameters  $r_b, \rho_c$  and  $\mu$  by equations (26) and (27).

The two pressures can similarly be given in terms of the above variables. The radial pressure can be computed from the second Einstein equation (5b) in a straightforward manner to yield

$$\kappa p_r(r) = \frac{2\kappa\rho_c}{3} - \frac{4\kappa\rho_c\mu r^2}{5r_b^2} - \kappa\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + \left( \frac{\kappa\rho_c}{3} - \frac{\kappa\rho_c\mu}{5} \right) \times \dots$$

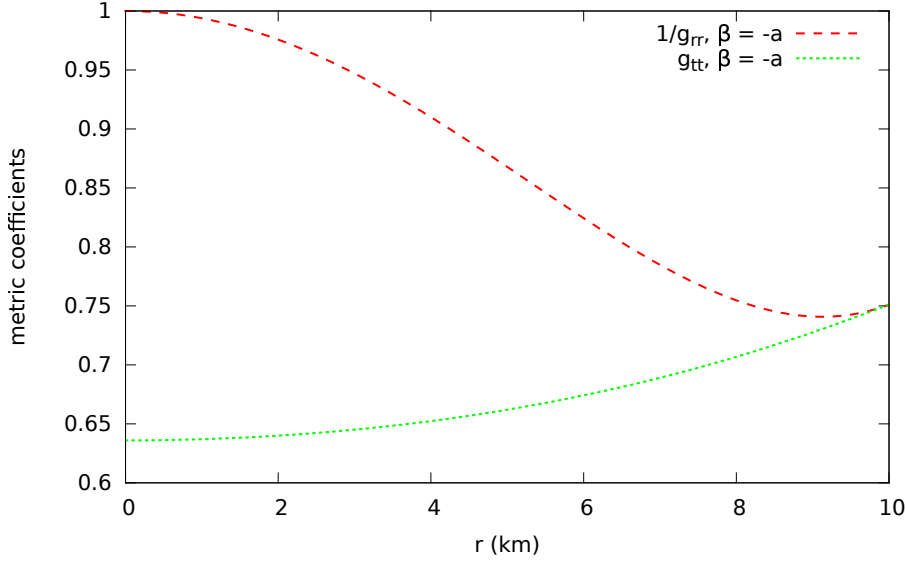


Fig. 2: Application of the boundary conditions that match the value and radial derivative of the metric function at  $r = r_b$  for the  $\phi^2 = 0$  case. The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and are chosen to give the conditions that might exist for a physically acceptable compact star. The parameter  $\mu = 1$  is chosen for simplicity.

$$\times \frac{\sqrt{1 - \frac{\kappa \rho_c}{3} r^2 + \frac{\kappa \mu \rho_c}{5 r_b^2} r^4}}{\gamma + \frac{2 \alpha r_b}{\sqrt{\kappa \rho_c \mu / 5}} \left[ \operatorname{arccoth} \left( \frac{1 - \sqrt{Z(r)}}{r^2 \sqrt{\frac{\kappa \rho_c \mu}{5 r_b^2}}} \right) - \operatorname{arccoth} \left( \frac{1 - \gamma}{r_b \sqrt{\kappa \rho_c \mu / 5}} \right) \right]}, \quad (29)$$

and similarly the tangential pressure is easily written in terms of the above as

$$p_{\perp}(r) = p_r - \beta r^2 = p_r + \frac{\kappa \rho_c \mu}{5 r_b^2} r^2. \quad (30)$$

This completes the solution, since all the metric functions and matter variables have been found in terms of the parameters in the ansatz and the radial coordinate  $r$ . If an equation of state for this solution is required, one could invert the density relation (17), to get an expression for  $r$  in terms of  $\rho$ , such that  $r = r_b \sqrt{(1 - \rho / \rho_c) / \mu}$ . Simple substitution in the expressions for the pressures (29) and (30) will then give the equation of state for both pressures  $p_t(\rho)$ , and  $p_{\perp}(\rho)$ , in a process similar to what was done in Ref [13].

### 3.2 The $\phi^2 > 0$ case

When  $\phi^2 > 0$ ,  $a + \beta > 0$ , which implies that  $\beta > -a$ . Since  $a$  has a specific dependence on the parameters  $\rho_c$ ,  $\mu$ , and  $r_b$ , one obtains  $\beta > -\frac{\kappa \mu \rho_c}{5 r_b^2}$ , which

allows  $\beta$  to have negative values, since the fraction in the last expression is positive definite.  $Y$  is then given in terms of the trigonometric functions given in Table 1, from which one can write expressions for the derivative of  $\frac{dY}{d\xi}$  by direct computation. Applying this boundary condition leads to the following relation among the constants of integration and the free parameters that describe the solution:

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = \phi [c_2 \cos(\phi\xi_b) - c_1 \sin(\phi\xi_b)] = \alpha := \frac{\kappa\rho_c(5-3\mu)}{60}.$$

Furthermore the boundary condition applied to  $Y$  leads to:

$$Y|_{\xi=\xi_b} = c_2 \sin(\phi\xi_b) + c_1 \cos(\phi\xi_b) = \gamma := \sqrt{1 + \frac{\kappa\rho_c r_b^2(3\mu-5)}{15}}.$$

These two equations provide a means for solving for  $c_1$  and  $c_2$ , to obtain

$$\begin{aligned} c_2 &= \gamma \sin(\phi\xi_b) + \frac{\alpha}{\phi} \cos(\phi\xi_b) \\ c_1 &= \gamma \cos(\phi\xi_b) - \frac{\alpha}{\phi} \sin(\phi\xi_b). \end{aligned}$$

A plot of the metric functions is shown in Figure 3 and shows how the metric functions and their radial derivatives match at the boundary  $r = r_b$ . The complete solution for the  $Y$ -metric function in this case is

$$\begin{aligned} Y(r) &= \\ &\left[ \gamma \cos(\phi\xi_b) - \frac{\alpha}{\phi} \sin(\phi\xi_b) \right] \cos \left[ \frac{2\phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right] + \\ &+ \left[ \gamma \sin(\phi\xi_b) + \frac{\alpha}{\phi} \cos(\phi\xi_b) \right] \sin \left[ \frac{2\phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right], \end{aligned} \quad (31)$$

which then leads to explicit expressions for the pressures  $p_\perp$  and  $p_r$  as

$$\begin{aligned} \kappa p_r(r) &= \frac{2\kappa\rho_c}{3} - \frac{4\kappa\rho_c\mu r^2}{5r_b^2} - \kappa\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4\phi \sqrt{1 - br^2 + ar^4} \times \\ &\times \frac{\left[ \gamma \sin(\phi\xi_b) + \frac{\alpha}{\phi} \cos(\phi\xi_b) \right] \cos(\phi\xi) - \left[ \gamma \cos(\phi\xi_b) - \frac{\alpha}{\phi} \sin(\phi\xi_b) \right] \sin(\phi\xi)}{\left[ \gamma \sin(\phi\xi_b) + \frac{\alpha}{\phi} \cos(\phi\xi_b) \right] \sin(\phi\xi) + \left[ \gamma \cos(\phi\xi_b) - \frac{\alpha}{\phi} \sin(\phi\xi_b) \right] \cos(\phi\xi)}, \end{aligned} \quad (32)$$

and

$$p_\perp(r) = p_r - \beta r^2. \quad (33)$$

As in the previous examples, and in particular the Tolman VII solution, inversion of the density–radial coordinate relation generates an equation of state, if required.

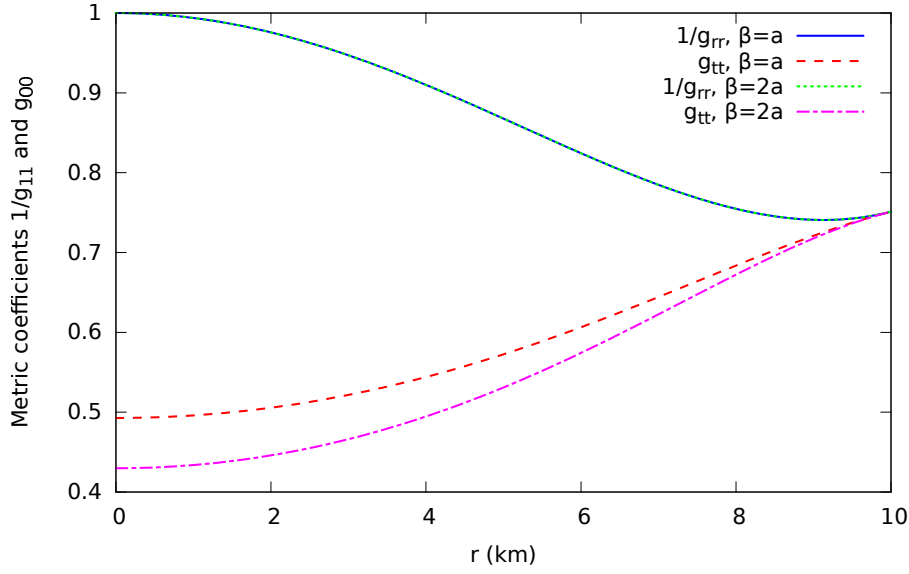


Fig. 3: Application of the boundary conditions that match the value and radial derivative of the metric function at  $r = r_b$  for the  $\phi^2 > 0$  case. The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and  $\mu = 1$ , with  $\beta$  given in the legend. Again the parameters are chosen such that the solution is applicable to a model of a compact star.

### 3.3 The $\phi^2 < 0$ case

When  $\phi^2 < 0$ ,  $a + \beta < 0$ , which implies that  $\beta < -a$ . Again using the explicit expression for  $a, \beta < -\frac{\kappa\mu\rho_c}{5r_b^2}$ , which forces  $\beta$  to have negative values, since the fraction in the last expression is positive definite. The expression for the derivative of  $Y$  can be found by direct computation. Applying the boundary condition on  $\frac{dY}{d\xi}$  at  $\xi_b$  leads to,

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = \phi [c_2 \cosh(\phi\xi_b) + c_1 \sinh(\phi\xi_b)] = \alpha,$$

while a similar calculation for  $Y$  at  $\xi_b$  yields

$$Y|_{\xi=\xi_b} = \gamma \Rightarrow c_2 \sinh(\phi\xi_b) + c_1 \cosh(\phi\xi_b) = \gamma.$$

These two equations provide explicit expressions for the integration constants,

$$\begin{aligned} c_2 &= \frac{\alpha}{\phi} \cosh(\phi\xi_b) - \gamma \sinh(\phi\xi_b) \\ c_1 &= \gamma \cosh(\phi\xi_b) - \frac{\alpha}{\phi} \sinh(\phi\xi_b), \end{aligned}$$

where  $\alpha$  and  $\gamma$  have the same expressions given in equation (26) and (27).

Plotting the metric functions as a function of  $r$ , Figure 4 again shows the result of matching the metric functions and their derivatives to exterior Schwarzschild solution at the boundary  $r = r_b$ .

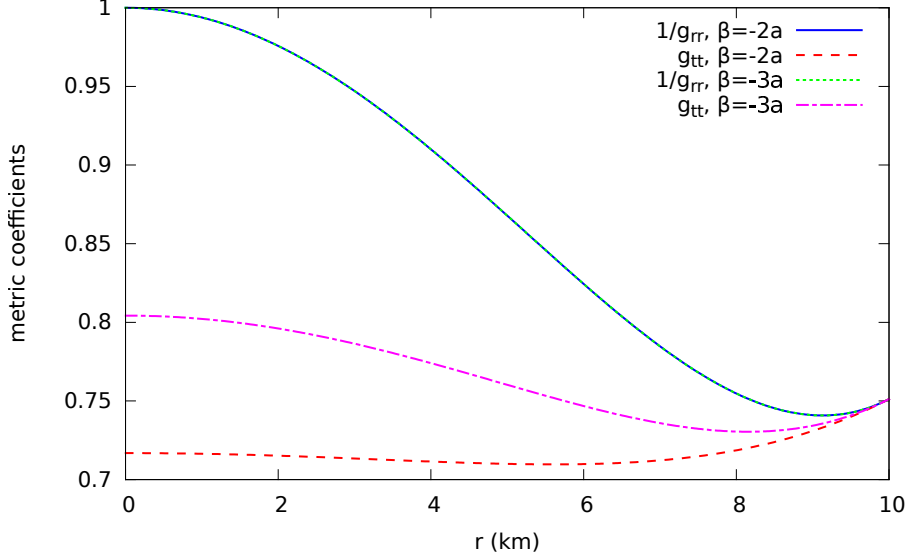


Fig. 4: Application of the boundary conditions that match the value and radial derivative of the metric function at  $r = r_b$ , for the  $\phi < 0$  case. The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and  $\mu = 1$ , with  $\beta$  given in the legend. Again the values for the constants have been chosen to generate models of stars that could be used for compact objects.

The complete solution for the  $Y$ -metric function in this case is

$$\begin{aligned}
 Y(r) = & \left[ \gamma \cosh(\phi \xi_b) - \frac{\alpha}{\phi} \sinh(\phi \xi_b) \right] \cosh \left[ \frac{2\phi}{\sqrt{a}} \operatorname{arccoth} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right] + \\
 & + \left[ \frac{\alpha}{\phi} \cosh(\phi \xi_b) - \gamma \sinh(\phi \xi_b) \right] \sinh \left[ \frac{2\phi}{\sqrt{a}} \operatorname{arccoth} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right], \quad (34)
 \end{aligned}$$

which then allows the matter variable  $p_r$  to be written as

$$\kappa p_r(r) = \frac{2\kappa\rho_c}{3} - \frac{4\kappa\rho_c\mu r^2}{5r_b^2} - \kappa\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4\phi \sqrt{1 - br^2 + ar^4} \times$$



$$\times \frac{\left[ \frac{\alpha}{\phi} \cosh(\phi \xi_b) - \gamma \sinh(\phi \xi_b) \right] \cosh(\phi \xi) + \left[ \gamma \cosh(\phi \xi_b) - \frac{\alpha}{\phi} \sinh(\phi \xi_b) \right] \sinh(\phi \xi)}{\left[ \gamma \cosh(\phi \xi_b) - \frac{\alpha}{\phi} \sinh(\phi \xi_b) \right] \cosh(\phi \xi) + \left[ \gamma \cosh(\phi \xi_b) - \frac{\alpha}{\phi} \sinh(\phi \xi_b) \right] \sinh(\phi \xi)}, \quad (35)$$

and  $p_\perp$ , the tangential pressure is expressed through the above as

$$p_\perp(r) = p_r - \beta r^2. \quad (36)$$

As with the previous examples, an explicit equation of state can be obtained from inverting equation (17) to obtain  $r(\rho)$ .

In the next section charged generalizations of solutions with anisotropic pressures will be discussed.

#### 4 Charged case with anisotropic pressures

In this section electrically charged solutions with  $q = kr^3 \neq 0$  are investigated. The simplest case sets  $\beta = 0$ , which eliminates the anisotropic pressure. This results in

$$4\Phi^2 = a - 2k^2.$$

Setting  $k = 0$  leads to the Tolman VII solution as expected. The general case with  $k \neq 0$  yields the solution given by Kyle and Martin [9].

Now consider the  $\Delta \neq 0$ , case instead. Requiring that  $\frac{\Delta}{x} = \frac{2q^2}{x^3}$  effectively “anisotropises” the electric charge allowing the latter to contribute to the anisotropy only, and considerably simplifies the solution to  $Y$ . This solution is examined in Section 4.1. If instead  $\Delta = \beta x$ , and  $2q^2 = 2k^2 x^3$ , one gets  $4\Phi^2 = a + \beta - 2k^2$ , which allows an analysis very similar to what was done in the previous Sections since  $\Phi^2$  can then be of either sign. This possibility is investigated in Sections 4.2 to 4.4

##### 4.1 “Anisotropised charge”

In this section solutions to the EMS where the electric charge and anisotropic pressure are related to each other through the relation  $\Delta = 2(q/x)^2$ , are discussed. From the arguments in Section (2.3),  $q = kr^3$ . This particular choice simplifies the differential equation for the  $Y$  metric function allowing for a solution analogous to the Tolman VII solution for  $Y$  to be written in the form

$$Y(\xi) = c_1 \cos(\Phi \xi) + c_2 \sin(\Phi \xi), \quad \text{with } \Phi = \sqrt{\frac{a}{4}}. \quad (37)$$

However this solution is fundamentally different from the Tolman VII solution which was a solution to the Einstein’s system of equation and not the Einstein–Maxwell system. There are a number of reasons for this:

1. The charge in this system is non-zero, unlike the Tolman VII solution, where  $Q = 0$ .
2. The presence of the anisotropic pressure in the solution means that  $p_\perp$  is *not* the same as the radial pressure  $p_r$ . This is clear since  $\Delta \neq 0$ .
3. Also, this solution will have to be matched to the Reissner-Nordström metric outside the sphere, as opposed to the Schwarzschild solution for the Tolman VII solution.

If these conditions are implemented, a fully-fledged new solution to the EMS can be obtained. Applying the boundary conditions (24) and (25), at the vacuum-matter interface leads to:

- The condition on the derivative

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = \Phi [c_2 \cos(\Phi\xi_b) - c_1 \sin(\Phi\xi_b)] = \alpha,$$

which can be rearranged to yield an equation for  $c_1$  and  $c_2$  in terms of previously defined constants:  $c_2 \cos(\Phi\xi) - c_1 \sin(\Phi\xi) = \alpha/\Phi$  and

- The second condition on the function  $Y$ :

$$Y(r_b) = c_1 \cos(\Phi\xi_b) + c_2 \sin(\Phi\xi_b) = \gamma,$$

Together this pair of equations can be solved for  $c_1$  and  $c_2$  to give

$$\begin{aligned} c_2 &= \gamma \sin(\Phi\xi_b) + \frac{\alpha}{\Phi} \cos(\Phi\xi_b) \\ c_1 &= \gamma \cos(\Phi\xi_b) - \frac{\alpha}{\Phi} \sin(\Phi\xi_b), \end{aligned}$$

Figure 5 which plots the metric functions demonstrates the effects of the parameter  $k$  which measures the magnitude of the electrostatic charge. For large values of  $k$  the behaviour of the metric functions deviates significantly from the  $k = 0$  cases studied in Section 3

The complete solution where the anisotropic pressures and the charge compensate for each other thus becomes

$$\begin{aligned} Y(r) = & \left( \gamma \cos(\Phi\xi_b) - \frac{\alpha}{\Phi} \sin(\Phi\xi_b) \right) \cos \left( \frac{2\Phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right) + \\ & + \left( \gamma \sin(\Phi\xi_b) + \frac{\alpha}{\Phi} \cos(\Phi\xi_b) \right) \sin \left( \frac{2\Phi}{\sqrt{a}} \coth^{-1} \left( \frac{1 + \sqrt{1 - br^2 + ar^4}}{r^2 \sqrt{a}} \right) \right), \end{aligned} \quad (38)$$

which then allows us to write the matter variables  $p_\perp$  and  $p_r$  as

$$\kappa p_r(r) = \frac{2\kappa\rho_c}{3} - \frac{4}{5} \left( \frac{\kappa\rho_c\mu}{r_b^2} - k^2 \right) r^2 - \kappa\rho_c \left[ 1 - \mu \left( \frac{r}{r_b} \right)^2 \right] + 4\Phi \sqrt{1 - br^2 + ar^4} \times$$

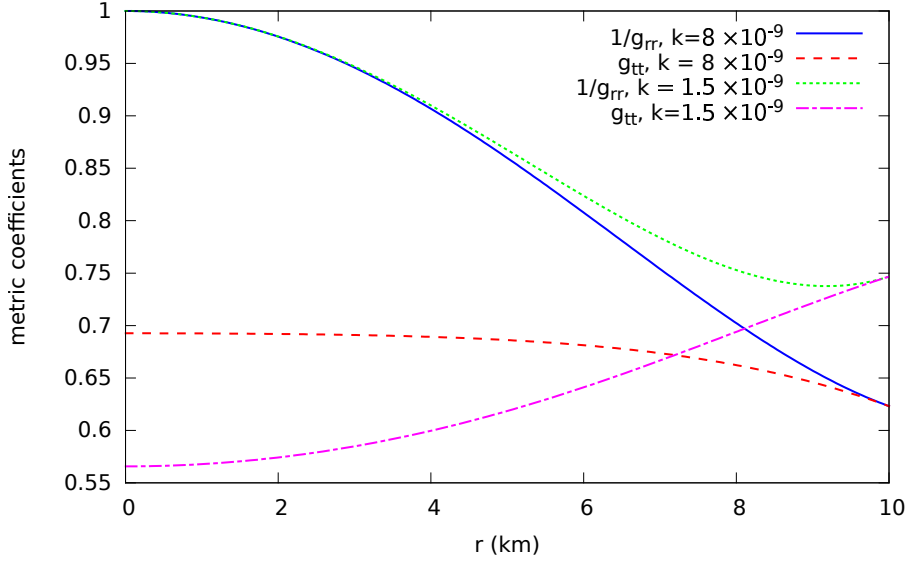


Fig. 5: Application of the boundary conditions that matches the value and radial derivative of the metric function at  $r = r_b$  for  $\Phi \neq 0$ , but where anisotropy compensates the charge. The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and  $\mu = 1$ , and have been chosen to mimic values thought to be present in actual compact stars.

$$\times \left\{ \frac{[\gamma \sin(\Phi \xi_b) + \frac{\alpha}{\Phi} \cos(\Phi \xi_b)] \cos(\Phi \xi) - [\gamma \cos(\Phi \xi_b) - \frac{\alpha}{\Phi} \sin(\Phi \xi_b)] \sin(\Phi \xi)}{[\gamma \sin(\Phi \xi_b) + \frac{\alpha}{\Phi} \cos(\Phi \xi_b)] \sin(\Phi \xi) + [\gamma \cos(\Phi \xi_b) - \frac{\alpha}{\Phi} \sin(\Phi \xi_b)] \cos(\Phi \xi)} \right\}, \quad (39)$$

and

$$p_{\perp}(r) = p_r - \Delta = p_r - 2k^2 r^2 \quad (40)$$

The intermediate variables in the above expressions for this case can be given in terms of the free parameters by :

$$\alpha = \frac{(\kappa \rho_c (5 - 3\mu) - 12k^2 r_b^2)}{60}, \quad \Delta(r) = 2k^2 r^2 = \frac{qr}{2k},$$

$$\gamma = \sqrt{1 + \frac{\kappa \rho_c r_b^2 (3\mu - 5)}{15} - \frac{k^2 r_b^2}{5}}, \quad \Phi^2 = \frac{1}{4} \left( \frac{\kappa \rho_c \mu}{r_b^2} - k^2 \right),$$

which completes the solution. As can be seen the solution could be expressed in terms of  $q$ , or  $\Delta$  exclusively, since these two functions are not independent in this particular solution. As with the previous examples, an explicit equation of state could be obtained since it requires a straightforward inversion of the density function's dependence on  $r$ . The total mass and charge of the object

modelled by this solution is obtained through (15), and for this particular case, these equations simplify to

$$M = 4\pi\rho_c r_b^3 \left( \frac{1}{3} - \frac{\mu}{5} \right) + \frac{k^2 r_b^5}{10}, \quad \text{and,} \quad Q = k r_b^3. \quad (41)$$

The last equation can be used to determine the charge density  $\sigma(r)$ , since from (15),

$$\int_0^{r_b} \bar{r}^2 \, d\bar{r} \left[ 4\pi\sigma(\bar{r})\sqrt{Z(\bar{r})} \right] = Q = k r_b^3 = \int_0^{r_b} \bar{r}^2 \, d\bar{r} [3k].$$

Direct comparison of terms yields the charge density

$$\sigma(r) = \frac{3k}{4\pi\sqrt{Z(r)}}. \quad (42)$$

This completes the solution for this case. Turning to the case where both charge and anisotropic pressure exist independently of each other, requires a thorough analysis of the different combinations of charge and anisotropic pressure, and how these conspire to change the character of the differential equation.

#### 4.2 The $\Phi^2 = 0$ case

If  $\Phi^2 = 0$  in equation (20), then  $2k^2 = a + \beta$ , and the solution for  $Y$  is simply  $Y = c_1 + c_2\xi$ , with  $c_1$  and  $c_2$  constants of integration. Applying boundary conditions on this solution then results in

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = c_2 = \alpha := \frac{1}{4} \left( \frac{\kappa\rho_c}{3} - \frac{3\kappa\rho_c\mu}{11} - \frac{4r_b^2\beta}{11} \right),$$

and

$$Y|_{\xi=\xi_b} = c_1 + c_2\xi_b = \gamma := \sqrt{1 + r_b^2\kappa\rho_b \left( \frac{2\mu}{11} - \frac{1}{3} \right) - \frac{\beta r_b^4}{11}},$$

which can be solved together algebraically to give the value of  $c_1$ . This completes the solution for  $Y$  in this particular case.

The  $Z$  metric function is still fixed by the Tolman assumption:  $Z = 1 - br^2 + ar^4$ . In this particular case  $a$  and  $b$  are given by:

$$a = \frac{2}{11} \left( \frac{\kappa\mu\rho_c}{r_b^2} - \frac{\beta}{2} \right), \quad \text{and} \quad b = \frac{\kappa\rho_c}{3}.$$

Given the relation  $a + \beta = 2k$ , only two parameters are arbitrary to completely specify the solution, and here,  $\beta$  is chosen in preference to  $k$ . This feature, and the consistent matching of the boundaries is shown in Figure 6.

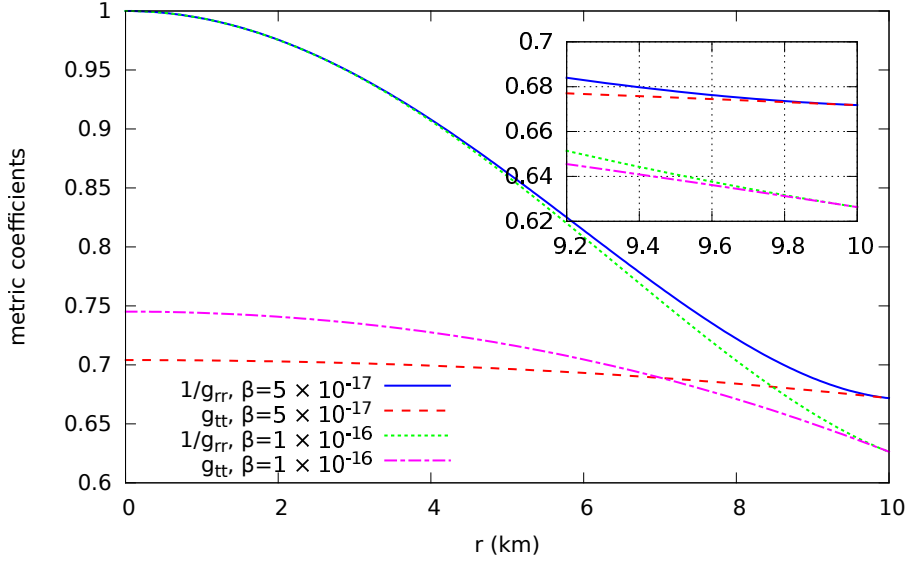


Fig. 6: Application of the boundary conditions that match in the value and radial derivative of the metric function at  $r = r_b$  for the  $\Phi^2 = 0$  case. The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and  $\mu = 1$ . The parameter values are chosen to mimic the values these parameters would have had in compact real stars.

Once the two metric functions are found, all other quantities are determined, in particular the radial pressure  $p_r$  is given by

$$p_r = \frac{1}{\kappa} \left[ \frac{4c_2 \sqrt{1 - br^2 + ar^4}}{c_1 + c_2 \xi} + 2b - 4ar^2 \right] - \rho(r),$$

and the tangential pressure  $p_\perp$ , in turn is  $p_\perp = p_r - \Delta/\kappa$ , giving

$$p_\perp = p_r - \frac{\beta x}{\kappa}.$$

The mass  $M$  and charge  $Q$  seen from the exterior, which are given by (41) result in

$$M = 4\pi\rho_c r_b^3 \left( \frac{1}{3} - \frac{7\mu}{55} \right) + \frac{\beta r_b^5}{22}, \quad \text{and,} \quad Q = r_b^3 \sqrt{\left( \frac{5\beta}{11} + \frac{\kappa\mu\rho_c}{11r_b^2} \right)}.$$

This completes the solution for this particular specific case.

### 4.3 The $\Phi^2 < 0$ case

Here  $(a + \beta - 2k^2)/4 < 0$ , and the solutions for  $Y$  can be written in the form

$$Y = c_1 \cosh(\Phi\xi) + c_2 \sinh(\Phi\xi).$$

In this particular case, one will not have a simplification wherein the charge could be compensated completely by the anisotropic pressure or mass, and one is then forced to deal with all three contributions. In this case  $2k^2 > a + \beta$ , and this leads to the belief that such a solution has very little chance of being physical, since the charge contribution dominates the mass and pressure.

Applying boundary conditions to this solution to obtain the values of the constants  $c_1$  and  $c_2$  through the same procedure as previously, leads to

$$\left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = \Phi [c_2 \cosh(\Phi\xi_b) + c_1 \sinh(\Phi\xi_b)] = \alpha,$$

and

$$Y(\xi_b) = c_2 \sinh(\Phi\xi_b) + c_1 \cosh(\Phi\xi_b) = \gamma.$$

Then using a procedure very similar to that of previous sections, the integration constants can be computed as

$$c_2 = \frac{\alpha}{\Phi} \cosh(\Phi\xi_b) - \gamma \sinh(\Phi\xi_b), \quad (43)$$

$$c_1 = \gamma \cosh(\Phi\xi_b) - \frac{\alpha}{\Phi} \sinh(\Phi\xi_b) \quad (44)$$

We show the solutions up to the boundary  $r = r_b$  in Figure 7, and note that in this case we need both  $\beta$  and  $k$  to completely specify one particular solution.

### 4.4 The $\Phi^2 > 0$ case

This implies  $(a + \beta - 2k^2)/4 > 0$ , and now the solutions are given by

$$Y = c_1 \cos(\Phi\xi) + c_2 \sin(\Phi\xi).$$

As in the previous case, the simplification wherein the charge could be compensated completely by the anisotropic pressure or mass, does not happen, and one is forced to deal with all three parameters. However the charge contribution will be less than the mass and pressure anisotropy contribution (since  $2k^2 < a + \beta$ ), therefore this would seem to be the most promising candidate for a new physically relevant solution. Investigating this solution and the remaining ones, in detail will allow conclusions about viability as a physical solution to be drawn.

Applying boundary conditions to this solution the values of the constants  $c_1$  and  $c_2$  are obtained.

$$1. \quad \left. \frac{dY}{d\xi} \right|_{\xi=\xi_b} = \Phi [c_2 \cos(\Phi\xi_b) - c_1 \sin(\Phi\xi_b)] = \alpha, \text{ and}$$

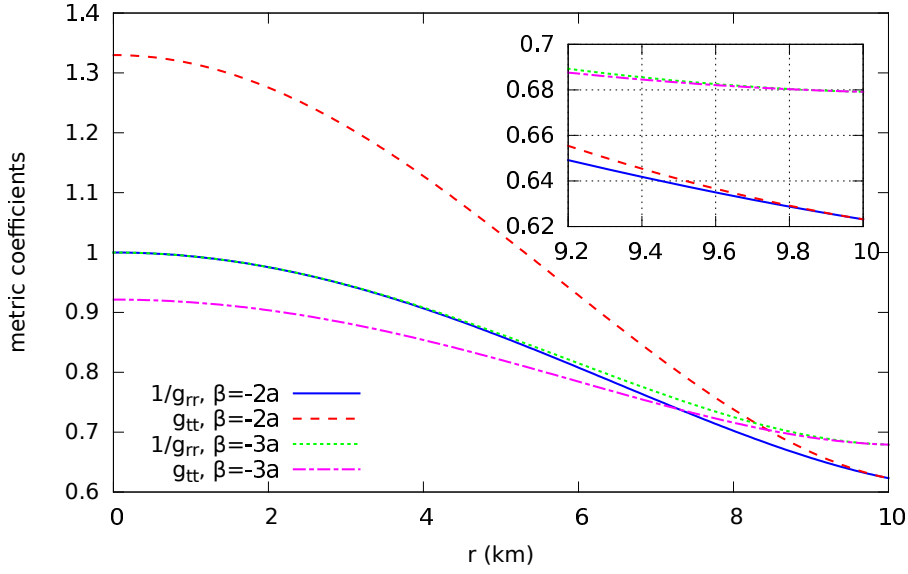


Fig. 7: Application of the boundary conditions that match the value and radial derivatives of the metric function at  $r = r_b$  for  $\Phi^2 < 0$ . The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and  $\mu = 1$ . The parameter values are chosen to mimic values in actual compact objects

$$2. Y(\xi_b) = c_2 \sin(\Phi \xi_b) + c_1 \cos(\Phi \xi_b) = \gamma.$$

Then using a procedure similar to that of previous sections one can obtain for the integration constants

$$c_2 = \gamma \sin(\Phi \xi_b) + \frac{\alpha}{\Phi} \cos(\Phi \xi_b), \quad (45)$$

$$c_1 = \gamma \cos(\Phi \xi_b) - \frac{\alpha}{\Phi} \sin(\Phi \xi_b). \quad (46)$$

The solution inside the star for different  $\beta$  and  $k$  are plotted in Figure 8.

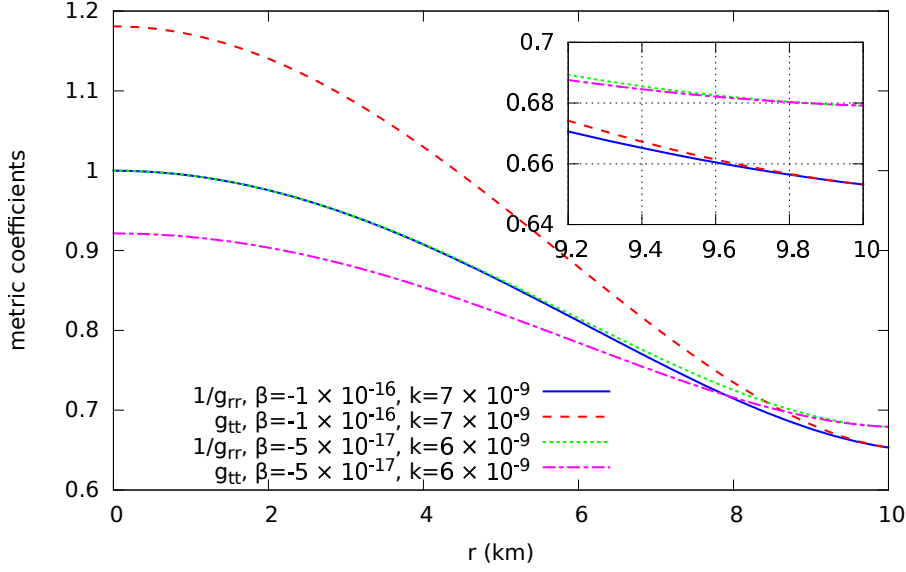


Fig. 8: Application of the boundary conditions that match the value and radial derivative of the metric function at  $r = r_b$  for  $\Phi^2 < 0$ . The parameter values are  $\rho_c = 1 \times 10^{18} \text{ kg} \cdot \text{m}^{-3}$ ,  $r_b = 1 \times 10^4 \text{ m}$  and  $\mu = 1$ , and were chosen to mimic the values in compact objects

## 5 Conclusions

Beginning with the Tolman VII ansatz that  $Z = 1 - br^2 + ar^4$ , together with the addition of electric charge and anisotropic pressures, it has been shown that a number of new solutions to the Einstein–Maxwell system of equations can be constructed. In addition these solutions are all regular in the radial coordinate  $r$ . The crucial step in constructing the new solutions is finding the function  $Y(r)$ , since the ODE leading to its solution is initially non-linear and second order.

The choices stemming from requiring a physical solution determine the radial dependence of  $Z$  and  $q$ . The functional form of  $Z$  is chosen so that the integral of  $Z^{-1/2}(x)$  can be computed analytically, and based on the Tolman VII solution, the mass density  $\rho$  generated by this assumption can be physically motivated. The form for  $q$ , as was argued in Section 2.3 was chosen to be in line with the assumption for  $Z$ , since any other choice would render the second order ODE [8] for  $Y$  to be in the form

$$(1 - bx + ax^2) \frac{d^2 Y}{dx^2} + \left( ax - \frac{b}{2} \right) \frac{dY}{dx} + \left( \frac{\Delta}{4x} + \frac{a}{4} - \frac{q^2}{2x^3} \right) Y = 0. \quad (47)$$

Indeed one can only realistically modify the choice of the anisotropic pressure in order to generate a new function. The crucial step in constructing solutions



stems from equation (20), which can then be converted by judicious choices to a situation where the second derivative of  $Y$  with respect to  $\xi$  is proportional to  $Y$ . The restriction of the radial dependence for the charge  $q$  to be a cubic function leads to an ODE for  $Y$  in the form:

$$\frac{d^2 Y}{d\xi^2} - \frac{k^2}{2} Y + \frac{Y}{4} \left( a + \frac{\Delta}{x} \right) = 0,$$

after the transformation to the  $\xi$  radial variable as discussed in Section 2.

Of course when pursuing uncharged solutions  $k$  (and  $q$ ), vanish. Then simple solutions can be obtained by an appropriate choice of  $\Delta$ . However, if charged solutions are required, modifying the terms in brackets in such a way as to keep a simple form for the  $Y$  ODE, appears to be the best way to proceed.

The application of boundary conditions on the metric functions  $Y$  and  $Z$  completed the closed form solutions in terms of parameters that can be *physically* interpreted. Expressions for the matter variables: the density  $\rho$ , the pressures  $p_r$  and  $p_\perp$ , the electric charge  $Q$ , and the mass  $M$  for the models were also specified. Issues such as the stability of the solutions to radial perturbations and/or the regularity of the matter variables have been investigated in [12] and will be published subsequently. The goal here is to provide a simple method of generating some solutions to the EMS that begin with physically motivated ansätze. Following a similar analysis given in [13] the expectation is that some of these solutions may lead to exact analytic models of realistic compact stars. This article should be regarded as the mathematical component of solution finding, and model building for the solutions. The physical description of compact objects described by these solutions will be forthcoming.

**Acknowledgements** We kindly thank Dr. Eugene Couch with whom we had discussions about the possibility of extending the main approach to new solutions. Funding for this research was obtained from a Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant.

## References

1. Bayin, S.S.: Solutions of Einstein's field equations for static fluid spheres. *Phys. Rev. D* **18**, 2745–2751 (1978). DOI 10.1103/PhysRevD.18.2745
2. Bowers, R.L., Liang, E.P.T.: Anisotropic Spheres in General Relativity. *ApJ* **188**, 657 (1974). DOI 10.1086/152760
3. Choquet-Bruhat, Y.: *General Relativity and the Einstein Equations*. Oxford Mathematical Monographs. OUP Oxford (2008)
4. Delgaty, M.S.R., Lake, K.: Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations. *Computer Physics Communications* **115**, 395–415 (1998). DOI 10.1016/S0010-4655(98)00130-1
5. D'Inverno, R.: *Introducing Einstein's Relativity*. Oxford University Press (1992)
6. Finch, M.R., Skea, J.E.F.: A review of the relativistic static sphere (1998). Unpublished, available at [www.dft.if.uerj.br/usuarios/JimSkea/papers/pfrev.ps](http://www.dft.if.uerj.br/usuarios/JimSkea/papers/pfrev.ps)
7. Herrera, L., Santos, N.: Local anisotropy in self-gravitating systems. *Physics Reports* **286**(2), 53 – 130 (1997). DOI 10.1016/S0370-1573(96)00042-7
8. Ivanov, B.V.: Static charged perfect fluid spheres in general relativity. *Physical Review D* **65**, 104,001 (2002)

9. Kyle, C., Martin, A.: Self-energy considerations in general relativity and the exact fields of charge and mass distributions. *Il Nuovo Cimento A* **50**(3), 583–604 (1967). DOI 10.1007/BF02823540
10. Letelier, P.S.: Anisotropic fluids with two-perfect-fluid components. *Phys. Rev. D* **22**, 807–813 (1980). DOI 10.1103/PhysRevD.22.807
11. Misner, C.W., Thorne, K.S., Wheeler, J.A.: *Gravitation*, 2 edn. W. H. Freeman (1973)
12. Raghoonundun, A.M.: Exact charged solutions with anisotropic pressures. Ph.D. thesis, University of Calgary (2016). (In preparation)
13. Raghoonundun, A.M., Hobill, D.W.: Possible physical realizations of the tolman vii solution. *Phys. Rev. D* **92**, 124,005 (2015). DOI 10.1103/PhysRevD.92.124005
14. Tolman, R.: *Relativity, Thermodynamics and Cosmology*. The international series of monographs on physics. Clarendon Press (1966)
15. Tolman, R.C.: Static Solutions of Einstein’s Field Equations for Spheres of Fluid. *Physical Review* **55**, 364–373 (1939). DOI 10.1103/PhysRev.55.364
16. Varela, V., Rahaman, F., Ray, S., Chakraborty, K., Kalam, M.: Charged anisotropic matter with linear or nonlinear equation of state. *Phys. Rev. D* **82**(4), 044052 (2010). DOI 10.1103/PhysRevD.82.044052
17. Wald, R.M.: *General Relativity*. University Of Chicago Press (1984)
18. Weyl, H.: *Space, Time, Matter*. Dover Books on Advanced Mathematics. Dover Publications (1952)